

Rotation of Axes

■ ■ For a discussion of conic sections, see Appendix B.

In precalculus or calculus you may have studied conic sections with equations of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

Here we show that the general second-degree equation

$$\mathbf{1} \quad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

can be analyzed by rotating the axes so as to eliminate the term Bxy .

In Figure 1 the x and y axes have been rotated about the origin through an acute angle θ to produce the X and Y axes. Thus, a given point P has coordinates (x, y) in the first coordinate system and (X, Y) in the new coordinate system. To see how X and Y are related to x and y we observe from Figure 2 that

$$\begin{aligned} X &= r \cos \phi & Y &= r \sin \phi \\ x &= r \cos(\theta + \phi) & y &= r \sin(\theta + \phi) \end{aligned}$$

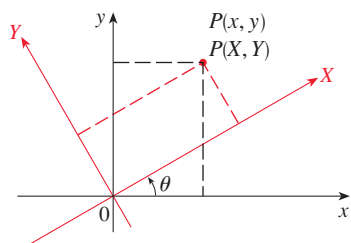


FIGURE 1

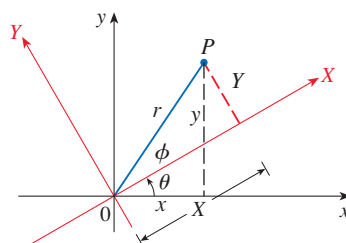


FIGURE 2

The addition formula for the cosine function then gives

$$\begin{aligned} x &= r \cos(\theta + \phi) = r(\cos \theta \cos \phi - \sin \theta \sin \phi) \\ &= (r \cos \phi) \cos \theta - (r \sin \phi) \sin \theta = X \cos \theta - Y \sin \theta \end{aligned}$$

A similar computation gives y in terms of X and Y and so we have the following formulas:

$$\mathbf{2} \quad x = X \cos \theta - Y \sin \theta \quad y = X \sin \theta + Y \cos \theta$$

By solving Equations 2 for X and Y we obtain

$$\mathbf{3} \quad X = x \cos \theta + y \sin \theta \quad Y = -x \sin \theta + y \cos \theta$$

EXAMPLE 1 If the axes are rotated through 60° , find the XY -coordinates of the point whose xy -coordinates are $(2, 6)$.

SOLUTION Using Equations 3 with $x = 2$, $y = 6$, and $\theta = 60^\circ$, we have

$$\begin{aligned} X &= 2 \cos 60^\circ + 6 \sin 60^\circ = 1 + 3\sqrt{3} \\ Y &= -2 \sin 60^\circ + 6 \cos 60^\circ = -\sqrt{3} + 3 \end{aligned}$$

The XY -coordinates are $(1 + 3\sqrt{3}, 3 - \sqrt{3})$. ■ ■

Now let's try to determine an angle θ such that the term Bxy in Equation 1 disappears when the axes are rotated through the angle θ . If we substitute from Equations 2 in Equation 1, we get

$$\begin{aligned} & A(X \cos \theta - Y \sin \theta)^2 + B(X \cos \theta - Y \sin \theta)(X \sin \theta + Y \cos \theta) \\ & + C(X \sin \theta + Y \cos \theta)^2 + D(X \cos \theta - Y \sin \theta) \\ & + E(X \sin \theta + Y \cos \theta) + F = 0 \end{aligned}$$

Expanding and collecting terms, we obtain an equation of the form

$$\boxed{4} \quad A'X^2 + B'XY + C'Y^2 + D'X + E'Y + F = 0$$

where the coefficient B' of XY is

$$\begin{aligned} B' &= 2(C - A) \sin \theta \cos \theta + B(\cos^2 \theta - \sin^2 \theta) \\ &= (C - A) \sin 2\theta + B \cos 2\theta \end{aligned}$$

To eliminate the XY term we choose θ so that $B' = 0$, that is,

$$(A - C) \sin 2\theta = B \cos 2\theta$$

or

$\boxed{5}$

$$\cot 2\theta = \frac{A - C}{B}$$

EXAMPLE 2 Show that the graph of the equation $xy = 1$ is a hyperbola.

SOLUTION Notice that the equation $xy = 1$ is in the form of Equation 1 where $A = 0$, $B = 1$, and $C = 0$. According to Equation 5, the xy term will be eliminated if we choose θ so that

$$\cot 2\theta = \frac{A - C}{B} = 0$$

This will be true if $2\theta = \pi/2$, that is, $\theta = \pi/4$. Then $\cos \theta = \sin \theta = 1/\sqrt{2}$ and Equations 2 become

$$x = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}} \quad y = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}$$

Substituting these expressions into the original equation gives

$$\left(\frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}\right)\left(\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}\right) = 1 \quad \text{or} \quad \frac{X^2}{2} - \frac{Y^2}{2} = 1$$

We recognize this as a hyperbola with vertices $(\pm\sqrt{2}, 0)$ in the XY -coordinate system. The asymptotes are $Y = \pm X$ in the XY -system, which correspond to the coordinate axes in the xy -system (see Figure 3). ■ ■

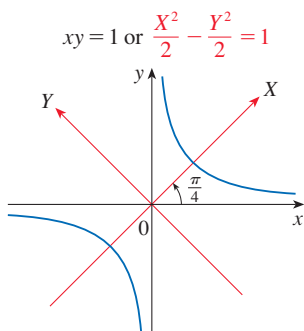


FIGURE 3

EXAMPLE 3 Identify and sketch the curve

$$73x^2 + 72xy + 52y^2 + 30x - 40y - 75 = 0$$

SOLUTION This equation is in the form of Equation 1 with $A = 73$, $B = 72$, and $C = 52$. Thus

$$\cot 2\theta = \frac{A - C}{B} = \frac{73 - 52}{72} = \frac{7}{24}$$

From the triangle in Figure 4 we see that

$$\cos 2\theta = \frac{7}{25}$$

The values of $\cos \theta$ and $\sin \theta$ can then be computed from the half-angle formulas:

$$\cos \theta = \sqrt{\frac{1 + \cos 2\theta}{2}} = \sqrt{\frac{1 + \frac{7}{25}}{2}} = \frac{4}{5}$$

$$\sin \theta = \sqrt{\frac{1 - \cos 2\theta}{2}} = \sqrt{\frac{1 - \frac{7}{25}}{2}} = \frac{3}{5}$$

The rotation equations (2) become

$$x = \frac{4}{5}X - \frac{3}{5}Y \quad y = \frac{3}{5}X + \frac{4}{5}Y$$

Substituting into the given equation, we have

$$\begin{aligned} 73\left(\frac{4}{5}X - \frac{3}{5}Y\right)^2 + 72\left(\frac{4}{5}X - \frac{3}{5}Y\right)\left(\frac{3}{5}X + \frac{4}{5}Y\right) + 52\left(\frac{3}{5}X + \frac{4}{5}Y\right)^2 \\ + 30\left(\frac{4}{5}X - \frac{3}{5}Y\right) - 40\left(\frac{3}{5}X + \frac{4}{5}Y\right) - 75 = 0 \end{aligned}$$

which simplifies to $4X^2 + Y^2 - 2Y = 3$

Completing the square gives

$$4X^2 + (Y - 1)^2 = 4 \quad \text{or} \quad X^2 + \frac{(Y - 1)^2}{4} = 1$$

and we recognize this as being an ellipse whose center is $(0, 1)$ in XY -coordinates. Since $\theta = \cos^{-1}\left(\frac{4}{5}\right) \approx 37^\circ$, we can sketch the graph in Figure 5.

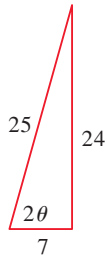


FIGURE 4

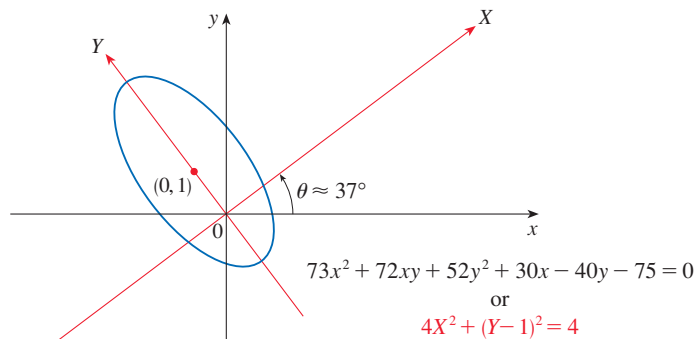


FIGURE 5

Exercises

A [Click here for answers.](#)

S [Click here for solutions.](#)

1–4 ■ Find the XY -coordinates of the given point if the axes are rotated through the specified angle.

1. $(1, 4)$, 30° 2. $(4, 3)$, 45°
 3. $(-2, 4)$, 60° 4. $(1, 1)$, 15°

5–12 ■ Use rotation of axes to identify and sketch the curve.

5. $x^2 - 2xy + y^2 - x - y = 0$
 6. $x^2 - xy + y^2 = 1$
 7. $x^2 + xy + y^2 = 1$
 8. $\sqrt{3}xy + y^2 = 1$
 9. $97x^2 + 192xy + 153y^2 = 225$
 10. $3x^2 - 12\sqrt{5}xy + 6y^2 + 9 = 0$
 11. $2\sqrt{3}xy - 2y^2 - \sqrt{3}x - y = 0$
 12. $16x^2 - 8\sqrt{2}xy + 2y^2 + (8\sqrt{2} - 3)x - (6\sqrt{2} + 4)y = 7$

13. (a) Use rotation of axes to show that the equation

$$36x^2 + 96xy + 64y^2 + 20x - 15y + 25 = 0$$

represents a parabola.

- (b) Find the XY -coordinates of the focus. Then find the xy -coordinates of the focus.

(c) Find an equation of the directrix in the xy -coordinate system.

14. (a) Use rotation of axes to show that the equation

$$2x^2 - 72xy + 23y^2 - 80x - 60y = 125$$

represents a hyperbola.

- (b) Find the XY -coordinates of the foci. Then find the xy -coordinates of the foci.
 (c) Find the xy -coordinates of the vertices.
 (d) Find the equations of the asymptotes in the xy -coordinate system.
 (e) Find the eccentricity of the hyperbola.
- 15.** Suppose that a rotation changes Equation 1 into Equation 4. Show that

$$A' + C' = A + C$$

16. Suppose that a rotation changes Equation 1 into Equation 4. Show that

$$(B')^2 - 4A'C' = B^2 - 4AC$$

17. Use Exercise 16 to show that Equation 1 represents (a) a parabola if $B^2 - 4AC = 0$, (b) an ellipse if $B^2 - 4AC < 0$, and (c) a hyperbola if $B^2 - 4AC > 0$, except in degenerate cases when it reduces to a point, a line, a pair of lines, or no graph at all.

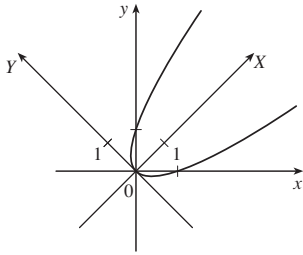
18. Use Exercise 17 to determine the type of curve in Exercises 9–12.

Answers
S [Click here for solutions.](#)

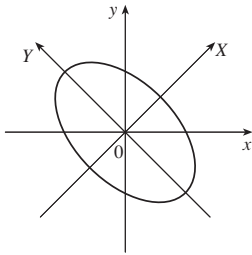
1. $((\sqrt{3} + 4)/2, (4\sqrt{3} - 1)/2)$

3. $(2\sqrt{3} - 1, \sqrt{3} + 2)$

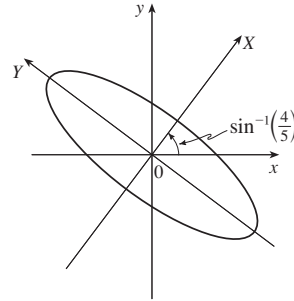
5. $X = \sqrt{2}Y^2$, parabola



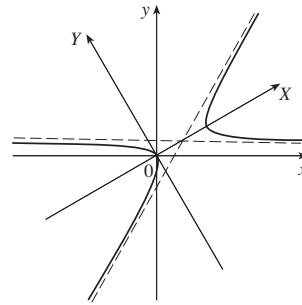
7. $3X^2 + Y^2 = 2$, ellipse



9. $X^2 + (Y^2/9) = 1$, ellipse



11. $(X - 1)^2 - 3Y^2 = 1$, hyperbola



13. (a) $Y - 1 = 4X^2$ (b) $(0, \frac{17}{16}), (-\frac{17}{20}, \frac{51}{80})$
 (c) $64x - 48y + 75 = 0$

Solutions: Rotation of Axes

$$1. X = 1 \cdot \cos 30^\circ + 4 \sin 30^\circ = 2 + \frac{\sqrt{3}}{2}, Y = -1 \cdot \sin 30^\circ + 4 \cos 30^\circ = 2\sqrt{3} - \frac{1}{2}.$$

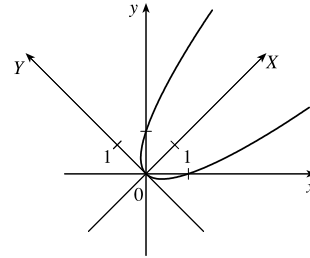
$$3. X = -2 \cos 60^\circ + 4 \sin 60^\circ = -1 + 2\sqrt{3}, Y = 2 \sin 60^\circ + 4 \cos 60^\circ = \sqrt{3} + 2.$$

$$5. \cot 2\theta = \frac{A-C}{B} = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Leftrightarrow \theta = \frac{\pi}{4} \Rightarrow \text{[by Equations 2]}$$

$$x = \frac{X-Y}{\sqrt{2}} \text{ and } y = \frac{X+Y}{\sqrt{2}}. \text{ Substituting these into the curve equation}$$

$$\text{gives } 0 = (x-y)^2 - (x+y) = 2Y^2 - \sqrt{2}X \text{ or } Y^2 = \frac{X}{\sqrt{2}}.$$

[Parabola, vertex $(0, 0)$, directrix $X = -1/(4\sqrt{2})$, focus $(1/(4\sqrt{2}), 0)$].



$$7. \cot 2\theta = \frac{A-C}{B} = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Leftrightarrow \theta = \frac{\pi}{4} \Rightarrow \text{[by}$$

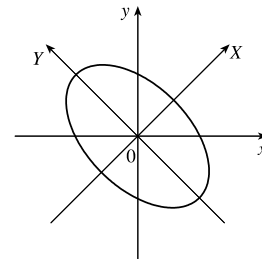
$$\text{Equations 2]} \quad x = \frac{X-Y}{\sqrt{2}} \text{ and } y = \frac{X+Y}{\sqrt{2}}. \text{ Substituting these into the}$$

curve equation gives

$$1 = \frac{X^2 - 2XY + Y^2}{2} + \frac{X^2 - Y^2}{2} + \frac{X^2 + 2XY + Y^2}{2} \Rightarrow$$

$$3X^2 + Y^2 = 2 \Rightarrow \frac{X^2}{2/3} + \frac{Y^2}{2} = 1. \text{ [An ellipse, center } (0, 0), \text{ foci on}$$

Y -axis with $a = \sqrt{2}$, $b = \sqrt{6}/3$, $c = 2\sqrt{3}/3$.]



$$9. \cot 2\theta = \frac{97-153}{192} = \frac{-7}{24} \Rightarrow \tan 2\theta = -\frac{24}{7} \Rightarrow \frac{\pi}{2} < 2\theta < \pi$$

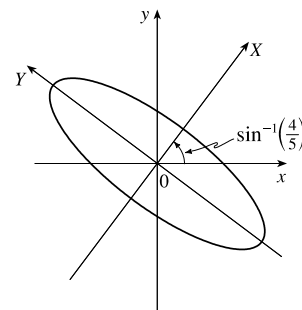
$$\text{and } \cos 2\theta = \frac{7}{25} \Rightarrow \frac{\pi}{4} < \theta < \frac{\pi}{2}, \cos \theta = \frac{3}{5}, \sin \theta = \frac{4}{5} \Rightarrow$$

$$x = X \cos \theta - Y \sin \theta = \frac{3X - 4Y}{5} \text{ and}$$

$$y = X \sin \theta + Y \cos \theta = \frac{4X + 3Y}{5}. \text{ Substituting, we get}$$

$$\frac{97}{25}(3X - 4Y)^2 + \frac{192}{25}(3X - 4Y)(4X + 3Y) + \frac{153}{25}(4X + 3Y)^2 = 225,$$

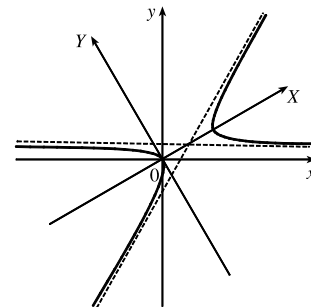
which simplifies to $X^2 + \frac{Y^2}{9} = 1$ (an ellipse with foci on Y -axis, centered at origin, $a = 3$, $b = 1$).



$$11. \cot 2\theta = \frac{A-C}{B} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6} \Rightarrow x = \frac{\sqrt{3}X - Y}{2},$$

$$y = \frac{X + \sqrt{3}Y}{2}. \text{ Substituting into the curve equation and simplifying gives}$$

$$4X^2 - 12Y^2 - 8X = 0 \Rightarrow (X-1)^2 - 3Y^2 = 1 \text{ [a hyperbola with foci on } X\text{-axis, centered at } (1, 0), a = 1, b = 1/\sqrt{3}, c = 2/\sqrt{3}].$$



13. (a) $\cot 2\theta = \frac{A-C}{B} = \frac{-7}{24}$ so, as in Exercise 9, $x = \frac{3X-4Y}{5}$ and $y = \frac{4X+3Y}{5}$.

Substituting and simplifying we get $100X^2 - 25Y + 25 = 0 \Rightarrow 4X^2 = Y - 1$, which is a parabola.

(b) The vertex is $(0, 1)$ and $p = \frac{1}{16}$, so the XY -coordinates of the focus are $(0, \frac{17}{16})$, and the xy -coordinates are

$$x = \frac{0 \cdot 3}{5} - \left(\frac{17}{16}\right)\left(\frac{4}{5}\right) = -\frac{17}{20} \text{ and } y = \frac{0 \cdot 4}{5} + \left(\frac{17}{16}\right)\left(\frac{3}{5}\right) = \frac{51}{80}.$$

(c) The directrix is $Y = \frac{15}{16}$, so $-x \cdot \frac{4}{5} + y \cdot \frac{3}{5} = \frac{15}{16} \Rightarrow 64x - 48y + 75 = 0$.

15. A rotation through θ changes Equation 1 to

$$A(X \cos \theta - Y \sin \theta)^2 + B(X \cos \theta - Y \sin \theta)(X \sin \theta + Y \cos \theta) + C(X \sin \theta + Y \cos \theta)^2 + D(X \cos \theta - Y \sin \theta) + E(X \sin \theta + Y \cos \theta) + F = 0.$$

Comparing this to Equation 4, we see that $A' + C' = A(\cos^2 \theta + \sin^2 \theta) + C(\sin^2 \theta + \cos^2 \theta) = A + C$.

17. Choose θ so that $B' = 0$. Then $B^2 - 4AC = (B')^2 - 4A'C' = -4A'C'$. But $A'C'$ will be 0 for a parabola, negative for a hyperbola (where the X^2 and Y^2 coefficients are of opposite sign), and positive for an ellipse (same sign for X^2 and Y^2 coefficients). So :

$$B^2 - 4AC = 0 \text{ for a parabola, } \quad B^2 - 4AC > 0 \text{ for a hyperbola, } \quad B^2 - 4AC < 0 \text{ for an ellipse.}$$

Note that the transformed equation takes the form $A'X^2 + C'Y^2 + D'X + E'Y + F = 0$, or by completing the square (assuming $A'C' \neq 0$), $A'(X')^2 + C'(Y')^2 = F'$, so that if $F' = 0$, the graph is either a pair of intersecting lines or a point, depending on the signs of A' and C' . If $F' \neq 0$ and $A'C' > 0$, then the graph is either an ellipse, a point, or nothing, and if $A'C' < 0$, the graph is a hyperbola. If A' or C' is 0, we cannot complete the square, so we get $A'(X')^2 + E'Y + F = 0$ or $C'(Y')^2 + D'X + F = 0$. This is a parabola, a straight line (if only the second-degree coefficient is nonzero), a pair of parallel lines (if the first-degree coefficient is zero and the other two have opposite signs), or an empty graph (if the first-degree coefficient is zero and the other two have the same sign).