Rotation of Axes

▪▪ For a discussion of conic sections, see Appendix B.

In precalculus or calculus you may have studied conic sections with equations of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

Here we show that the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{1}$$

can be analyzed by rotating the axes so as to eliminate the term Bxy.

In Figure 1 the x and y axes have been rotated about the origin through an acute angle θ to produce the X and Y axes. Thus, a given point P has coordinates (x, y) in the first coordinate system and (X, Y) in the new coordinate system. To see how X and Y are related to x and y we observe from Figure 2 that

$$X = r\cos\phi \qquad Y = r\sin\phi$$

$$x = r\cos(\theta + \phi) \qquad y = r\sin(\theta + \phi)$$

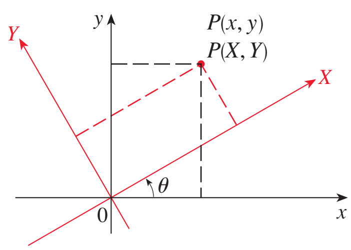


FIGURE 1

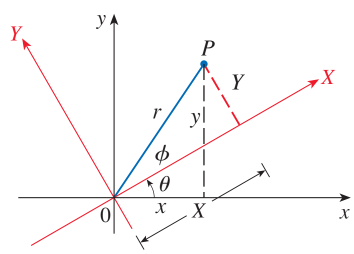


FIGURE 2

The addition formula for the cosine function then gives

$$x = r\cos(\theta + \phi) = r(\cos\theta\cos\phi - \sin\theta\sin\phi)$$

$$= (r\cos\phi)\cos\theta - (r\sin\phi)\sin\theta = X\cos\theta - Y\sin\theta$$

A similar computation gives y in terms of X and Y and so we have the following formulas:

$$x = X\cos\theta - Y\sin\theta \qquad y = X\sin\theta + Y\cos\theta \tag{2}$$

By solving Equations 2 for X and Y we obtain

$$X = x\cos\theta + y\sin\theta \qquad Y = -x\sin\theta + y\cos\theta \tag{3}$$

EXAMPLE 1 If the axes are rotated through 60°, find the XY-coordinates of the point whose xy-coordinates are (2, 6).

SOLUTION Using Equations 3 with $x = 2$, $y = 6$, and $\theta = 60°$, we have

$$X = 2\cos 60° + 6\sin 60° = 1 + 3\sqrt{3}$$

$$Y = -2\sin 60° + 6\cos 60° = -\sqrt{3} + 3$$

The XY-coordinates are $\left(1 + 3\sqrt{3}, 3 - \sqrt{3}\right)$. ▪▪

Now let's try to determine an angle θ such that the term Bxy in Equation 1 disappears when the axes are rotated through the angle θ. If we substitute from Equations 2 in Equation 1, we get

$$A(X\cos\theta - Y\sin\theta)^2 + B(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta)$$
$$+ C(X\sin\theta + Y\cos\theta)^2 + D(X\cos\theta - Y\sin\theta)$$
$$+ E(X\sin\theta + Y\cos\theta) + F = 0$$

Expanding and collecting terms, we obtain an equation of the form

4 $$A'X^2 + B'XY + C'Y^2 + D'X + E'Y + F = 0$$

where the coefficient B' of XY is

$$B' = 2(C - A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta)$$
$$= (C - A)\sin 2\theta + B\cos 2\theta$$

To eliminate the XY term we choose θ so that $B' = 0$, that is,

$$(A - C)\sin 2\theta = B\cos 2\theta$$

or

5 $$\boxed{\cot 2\theta = \frac{A - C}{B}}$$

EXAMPLE 2 Show that the graph of the equation $xy = 1$ is a hyperbola.

SOLUTION Notice that the equation $xy = 1$ is in the form of Equation 1 where $A = 0$, $B = 1$, and $C = 0$. According to Equation 5, the xy term will be eliminated if we choose θ so that

$$\cot 2\theta = \frac{A - C}{B} = 0$$

This will be true if $2\theta = \pi/2$, that is, $\theta = \pi/4$. Then $\cos\theta = \sin\theta = 1/\sqrt{2}$ and Equations 2 become

$$x = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}} \qquad y = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}$$

Substituting these expressions into the original equation gives

$$\left(\frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}\right)\left(\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}\right) = 1 \quad \text{or} \quad \frac{X^2}{2} - \frac{Y^2}{2} = 1$$

We recognize this as a hyperbola with vertices $(\pm\sqrt{2}, 0)$ in the XY-coordinate system. The asymptotes are $Y = \pm X$ in the XY-system, which correspond to the coordinate axes in the xy-system (see Figure 3). ■■

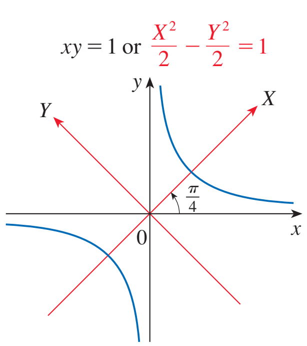


FIGURE 3

EXAMPLE 3 Identify and sketch the curve

$$73x^2 + 72xy + 52y^2 + 30x - 40y - 75 = 0$$

SOLUTION This equation is in the form of Equation 1 with $A = 73$, $B = 72$, and $C = 52$. Thus

$$\cot 2\theta = \frac{A - C}{B} = \frac{73 - 52}{72} = \frac{7}{24}$$

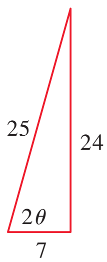


FIGURE 4

From the triangle in Figure 4 we see that

$$\cos 2\theta = \tfrac{7}{25}$$

The values of $\cos\theta$ and $\sin\theta$ can then be computed from the half-angle formulas:

$$\cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}} = \sqrt{\frac{1 + \frac{7}{25}}{2}} = \frac{4}{5}$$

$$\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}} = \sqrt{\frac{1 - \frac{7}{25}}{2}} = \frac{3}{5}$$

The rotation equations (2) become

$$x = \tfrac{4}{5}X - \tfrac{3}{5}Y \qquad y = \tfrac{3}{5}X + \tfrac{4}{5}Y$$

Substituting into the given equation, we have

$$73\left(\tfrac{4}{5}X - \tfrac{3}{5}Y\right)^2 + 72\left(\tfrac{4}{5}X - \tfrac{3}{5}Y\right)\left(\tfrac{3}{5}X + \tfrac{4}{5}Y\right) + 52\left(\tfrac{3}{5}X + \tfrac{4}{5}Y\right)^2$$

$$+ 30\left(\tfrac{4}{5}X - \tfrac{3}{5}Y\right) - 40\left(\tfrac{3}{5}X + \tfrac{4}{5}Y\right) - 75 = 0$$

which simplifies to $$4X^2 + Y^2 - 2Y = 3$$

Completing the square gives

$$4X^2 + (Y - 1)^2 = 4 \qquad \text{or} \qquad X^2 + \frac{(Y - 1)^2}{4} = 1$$

and we recognize this as being an ellipse whose center is $(0, 1)$ in XY-coordinates. Since $\theta = \cos^{-1}\left(\frac{4}{5}\right) \approx 37°$, we can sketch the graph in Figure 5.

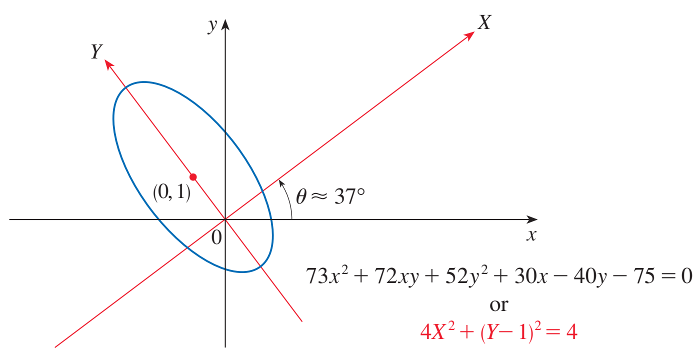


FIGURE 5

Exercises

A Click here for answers. S Click here for solutions.

1–4 ■ Find the XY-coordinates of the given point if the axes are rotated through the specified angle.

1. $(1, 4)$, $30°$

2. $(4, 3)$, $45°$

3. $(-2, 4)$, $60°$

4. $(1, 1)$, $15°$

5–12 ■ Use rotation of axes to identify and sketch the curve.

5. $x^2 - 2xy + y^2 - x - y = 0$

6. $x^2 - xy + y^2 = 1$

7. $x^2 + xy + y^2 = 1$

8. $\sqrt{3}xy + y^2 = 1$

9. $97x^2 + 192xy + 153y^2 = 225$

10. $3x^2 - 12\sqrt{5}xy + 6y^2 + 9 = 0$

11. $2\sqrt{3}xy - 2y^2 - \sqrt{3}x - y = 0$

12. $16x^2 - 8\sqrt{2}xy + 2y^2 + (8\sqrt{2} - 3)x - (6\sqrt{2} + 4)y = 7$

13. (a) Use rotation of axes to show that the equation

$$36x^2 + 96xy + 64y^2 + 20x - 15y + 25 = 0$$

represents a parabola.

(b) Find the XY-coordinates of the focus. Then find the xy-coordinates of the focus.

(c) Find an equation of the directrix in the xy-coordinate system.

14. (a) Use rotation of axes to show that the equation

$$2x^2 - 72xy + 23y^2 - 80x - 60y = 125$$

represents a hyperbola.

(b) Find the XY-coordinates of the foci. Then find the xy-coordinates of the foci.

(c) Find the xy-coordinates of the vertices.

(d) Find the equations of the asymptotes in the xy-coordinate system.

(e) Find the eccentricity of the hyperbola.

15. Suppose that a rotation changes Equation 1 into Equation 4. Show that

$$A' + C' = A + C$$

16. Suppose that a rotation changes Equation 1 into Equation 4. Show that

$$(B')^2 - 4A'C' = B^2 - 4AC$$

17. Use Exercise 16 to show that Equation 1 represents (a) a parabola if $B^2 - 4AC = 0$, (b) an ellipse if $B^2 - 4AC < 0$, and (c) a hyperbola if $B^2 - 4AC > 0$, except in degenerate cases when it reduces to a point, a line, a pair of lines, or no graph at all.

18. Use Exercise 17 to determine the type of curve in Exercises 9–12.

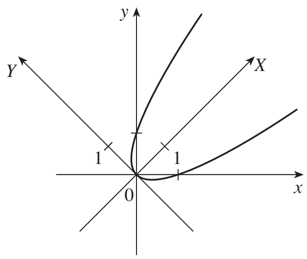
Answers

S Click here for solutions.

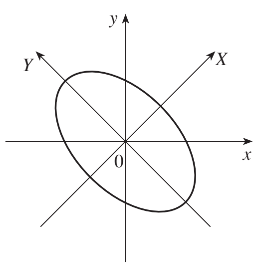
1. $((\sqrt{3}+4)/2, (4\sqrt{3}-1)/2)$

3. $(2\sqrt{3}-1, \sqrt{3}+2)$

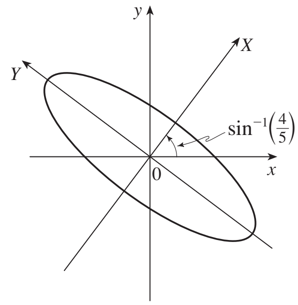
5. $X = \sqrt{2}Y^2$, parabola



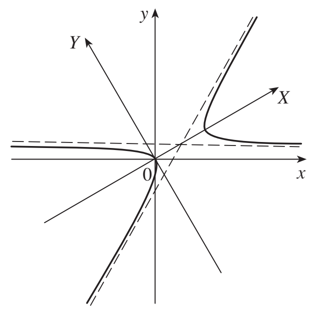
7. $3X^2 + Y^2 = 2$, ellipse



9. $X^2 + (Y^2/9) = 1$, ellipse



11. $(X-1)^2 - 3Y^2 = 1$, hyperbola



13. (a) $Y - 1 = 4X^2$ (b) $\left(0, \frac{17}{16}\right), \left(-\frac{17}{20}, \frac{51}{80}\right)$
(c) $64x - 48y + 75 = 0$

Solutions: Rotation of Axes

1. $X = 1 \cdot \cos 30^\circ + 4 \sin 30^\circ = 2 + \frac{\sqrt{3}}{2}$, $Y = -1 \cdot \sin 30^\circ + 4 \cos 30^\circ = 2\sqrt{3} - \frac{1}{2}$.

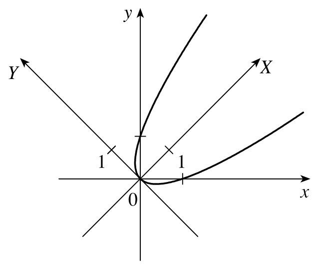
3. $X = -2 \cos 60^\circ + 4 \sin 60^\circ = -1 + 2\sqrt{3}$, $Y = 2 \sin 60^\circ + 4 \cos 60^\circ = \sqrt{3} + 2$.

5. $\cot 2\theta = \dfrac{A - C}{B} = 0 \quad \Rightarrow \quad 2\theta = \frac{\pi}{2} \quad \Leftrightarrow \quad \theta = \frac{\pi}{4} \quad \Rightarrow$ [by Equations 2]

$x = \dfrac{X - Y}{\sqrt{2}}$ and $y = \dfrac{X + Y}{\sqrt{2}}$. Substituting these into the curve equation

gives $0 = (x - y)^2 - (x + y) = 2Y^2 - \sqrt{2}X$ or $Y^2 = \dfrac{X}{\sqrt{2}}$.

[Parabola, vertex $(0, 0)$, directrix $X = -1/\left(4\sqrt{2}\right)$, focus $\left(1/\left(4\sqrt{2}\right), 0\right)$].



7. $\cot 2\theta = \dfrac{A - C}{B} = 0 \quad \Rightarrow \quad 2\theta = \frac{\pi}{2} \quad \Leftrightarrow \quad \theta = \frac{\pi}{4} \quad \Rightarrow$ [by

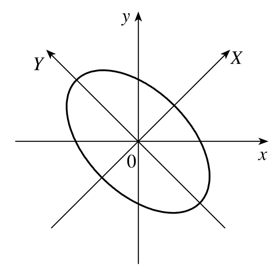
Equations 2] $\quad x = \dfrac{X - Y}{\sqrt{2}}$ and $y = \dfrac{X + Y}{\sqrt{2}}$. Substituting these into the

curve equation gives

$$1 = \frac{X^2 - 2XY + Y^2}{2} + \frac{X^2 - Y^2}{2} + \frac{X^2 + 2XY + Y^2}{2} \quad \Rightarrow$$

$3X^2 + Y^2 = 2 \quad \Rightarrow \quad \dfrac{X^2}{2/3} + \dfrac{Y^2}{2} = 1$. [An ellipse, center $(0, 0)$, foci on

Y-axis with $a = \sqrt{2}$, $b = \sqrt{6}/3$, $c = 2\sqrt{3}/3$.]



9. $\cot 2\theta = \dfrac{97 - 153}{192} = \dfrac{-7}{24} \quad \Rightarrow \quad \tan 2\theta = -\frac{24}{7} \quad \Rightarrow \quad \frac{\pi}{2} < 2\theta < \pi$

and $\cos 2\theta = \frac{-7}{25} \quad \Rightarrow \quad \frac{\pi}{4} < \theta < \frac{\pi}{2}$, $\cos\theta = \frac{3}{5}$, $\sin\theta = \frac{4}{5} \quad \Rightarrow$

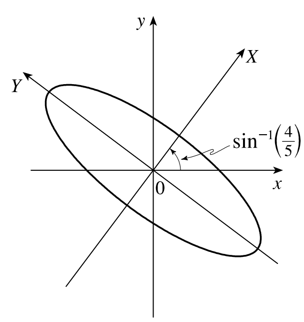
$x = X\cos\theta - Y\sin\theta = \dfrac{3X - 4Y}{5}$ and

$y = X\sin\theta + Y\cos\theta = \dfrac{4X + 3Y}{5}$. Substituting, we get

$\frac{97}{25}(3X - 4Y)^2 + \frac{192}{25}(3X - 4Y)(4X + 3Y) + \frac{153}{25}(4X + 3Y)^2 = 225$,

which simplifies to $X^2 + \dfrac{Y^2}{9} = 1$ (an ellipse with foci on Y-axis, centered

at origin, $a = 3$, $b = 1$).

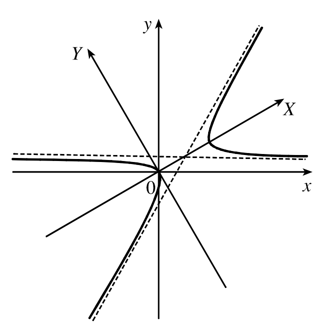


11. $\cot 2\theta = \dfrac{A - C}{B} = \dfrac{1}{\sqrt{3}} \quad \Rightarrow \quad \theta = \dfrac{\pi}{6} \quad \Rightarrow \quad x = \dfrac{\sqrt{3}X - Y}{2}$,

$y = \dfrac{X + \sqrt{3}Y}{2}$. Substituting into the curve equation and simplifying gives

$4X^2 - 12Y^2 - 8X = 0 \Rightarrow (X - 1)^2 - 3Y^2 = 1$ [a hyperbola with foci

on X-axis, centered at $(1, 0)$, $a = 1$, $b = 1/\sqrt{3}$, $c = 2/\sqrt{3}$].



13. (a) $\cot 2\theta = \dfrac{A-C}{B} = \dfrac{-7}{24}$ so, as in Exercise 9, $x = \dfrac{3X - 4Y}{5}$ and $y = \dfrac{4X + 3Y}{5}$.

Substituting and simplifying we get $100X^2 - 25Y + 25 = 0 \quad \Rightarrow \quad 4X^2 = Y - 1$, which is a parabola.

(b) The vertex is $(0, 1)$ and $p = \frac{1}{16}$, so the XY-coordinates of the focus are $\left(0, \frac{17}{16}\right)$, and the xy-coordinates are $x = \frac{0 \cdot 3}{5} - \left(\frac{17}{16}\right)\left(\frac{4}{5}\right) = -\frac{17}{20}$ and $y = \frac{0 \cdot 4}{5} + \left(\frac{17}{16}\right)\left(\frac{3}{5}\right) = \frac{51}{80}$.

(c) The directrix is $Y = \frac{15}{16}$, so $-x \cdot \frac{4}{5} + y \cdot \frac{3}{5} = \frac{15}{16} \quad \Rightarrow \quad 64x - 48y + 75 = 0$.

15. A rotation through θ changes Equation 1 to

$A(X\cos\theta - Y\sin\theta)^2 + B(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + C(X\sin\theta + Y\cos\theta)^2 + D(X\cos\theta - Y\sin\theta)$
$+ E(X\sin\theta + Y\cos\theta) + F = 0$.

Comparing this to Equation 4, we see that $A' + C' = A(\cos^2\theta + \sin^2\theta) + C(\sin^2\theta + \cos^2\theta) = A + C$.

17. Choose θ so that $B' = 0$. Then $B^2 - 4AC = (B')^2 - 4A'C' = -4A'C'$. But $A'C'$ will be 0 for a parabola, negative for a hyperbola (where the X^2 and Y^2 coefficients are of opposite sign), and positive for an ellipse (same sign for X^2 and Y^2 coefficients). So :

$B^2 - 4AC = 0$ for a parabola, $\qquad B^2 - 4AC > 0$ for a hyperbola, $\qquad B^2 - 4AC < 0$ for an ellipse.

Note that the transformed equation takes the form $A'X^2 + C'Y^2 + D'X + E'Y + F = 0$, or by completing the square (assuming $A'C' \neq 0$), $A'(X')^2 + C'(Y')^2 = F'$, so that if $F' = 0$, the graph is either a pair of intersecting lines or a point, depending on the signs of A' and C'. If $F' \neq 0$ and $A'C' > 0$, then the graph is either an ellipse, a point, or nothing, and if $A'C' < 0$, the graph is a hyperbola. If A' or C' is 0, we cannot complete the square, so we get $A'(X')^2 + E'Y + F = 0$ or $C'(Y')^2 + D'X + F' = 0$. This is a parabola, a straight line (if only the second-degree coefficient is nonzero), a pair of parallel lines (if the first-degree coefficient is zero and the other two have opposite signs), or an empty graph (if the first-degree coefficient is zero and the other two have the same sign).