CHALLENGE PROBLEMS

CHAPTER 4

A Click here for answers. S Click here for solutions.

1. If $x \sin \pi x = \int_0^{x^2} f(t)\,dt$, where f is a continuous function, find $f(4)$.

2. In this problem we approximate the sine function on the interval $[0, \pi]$ by three quadratic functions, each of which has the same zeros as the sine function on this interval.
(a) Find a quadratic function f such that $f(0) = f(\pi) = 0$ and which has the same maximum value as sin on $[0, \pi]$.
(b) Find a quadratic function g such that $g(0) = g(\pi) = 0$ and which has the same rate of change as the sine function at 0 and π.
(c) Find a quadratic function h such that $h(0) = h(\pi) = 0$ and the area under h from 0 to π is the same as for the sine function.
(d) Illustrate by graphing f, g, h, and the sine function in the same viewing rectangle $[0, \pi]$ by $[0, 1]$. Identify which graph belongs to each function.

3. Show that $\frac{1}{17} \leqslant \int_1^2 \frac{1}{1 + x^4}\,dx \leqslant \frac{7}{24}$.

4. Suppose the curve $y = f(x)$ passes through the origin and the point (1, 1). Find the value of the integral $\int_0^1 f'(x)\,dx$.

5. Find a function f such that $f(1) = -1$, $f(4) = 7$, and $f'(x) > 3$ for all x, or prove that such a function cannot exist.

6. (a) Graph several members of the family of functions $f(x) = (2cx - x^2)/c^3$ for $c > 0$ and look at the regions enclosed by these curves and the x-axis. Make a conjecture about how the areas of these regions are related.
(b) Prove your conjecture in part (a).
(c) Take another look at the graphs in part (a) and use them to sketch the curve traced out by the vertices (highest points) of the family of functions. Can you guess what kind of curve this is?
(d) Find the equation of the curve you sketched in part (c).

7. If $f(x) = \int_0^{g(x)} \frac{1}{\sqrt{1 + t^3}}\,dt$, where $g(x) = \int_0^{\cos x} [1 + \sin(t^2)]\,dt$, find $f'(\pi/2)$.

8. If $f(x) = \int_0^x x^2 \sin(t^2)\,dt$, find $f'(x)$.

9. Find the interval $[a, b]$ for which the value of the integral $\int_a^b (2 + x - x^2)\,dx$ is a maximum.

10. Use an integral to estimate the sum $\sum_{i=1}^{10000} \sqrt{i}$.

11. (a) Evaluate $\int_0^n [\![x]\!]\,dx$, where n is a positive integer.
(b) Evaluate $\int_a^b [\![x]\!]\,dx$, where a and b are real numbers with $0 \leqslant a < b$.

12. Find $\frac{d^2}{dx^2} \int_0^x \left(\int_1^{\sin t} \sqrt{1 + u^4}\,du \right) dt$.

13. If f is a differentiable function such that $\int_0^x f(t)\,dt = [f(x)]^2$ for all x, find f.

14. A circular disk of radius r is used in an evaporator and is rotated in a vertical plane. If it is to be partially submerged in the liquid so as to maximize the exposed wetted area of the disk, show that the center of the disk should be positioned at a height $r/\sqrt{1 + \pi^2}$ above the surface of the liquid.

15. Prove that if f is continuous, then $\int_0^x f(u)(x - u)\,du = \int_0^x \left(\int_0^u f(t)\,dt \right) du$.

16. The figure shows a region consisting of all points inside a square that are closer to the center than to the sides of the square. Find the area of the region.

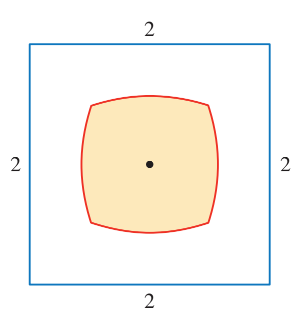

FIGURE FOR PROBLEM 16

17. Evaluate $\lim_{n \to \infty} \left(\frac{1}{\sqrt{n}\sqrt{n + 1}} + \frac{1}{\sqrt{n}\sqrt{n + 2}} + \cdots + \frac{1}{\sqrt{n}\sqrt{n + n}} \right)$.

ANSWERS

S Solutions

1. $\pi/2$ **5.** Does not exist **7.** -1 **9.** $[-1, 2]$

11. (a) $(n-1)n/2$ (b) $\frac{1}{2}[\![b]\!](2b - [\![b]\!] - 1) - \frac{1}{2}[\![a]\!](2a - [\![a]\!] - 1)$

13. $f(x) = \frac{1}{2}x$ or $f(x) = 0$ **17.** $2(\sqrt{2} - 1)$

SOLUTIONS

E Exercises

1. Differentiating both sides of the equation $x\sin\pi x = \int_0^{x^2} f(t)\,dt$ (using FTC1 and the Chain Rule for the right side) gives $\sin\pi x + \pi x\cos\pi x = 2xf\left(x^2\right)$. Letting $x = 2$ so that $f\left(x^2\right) = f(4)$, we obtain $\sin 2\pi + 2\pi\cos 2\pi = 4f(4)$, so $f(4) = \frac{1}{4}(0 + 2\pi\cdot 1) = \frac{\pi}{2}$.

3. For $1 \le x \le 2$, we have $x^4 \le 2^4 = 16$, so $1 + x^4 \le 17$ and $\dfrac{1}{1+x^4} \ge \dfrac{1}{17}$. Thus,

$\displaystyle\int_1^2 \frac{1}{1+x^4}\,dx \ge \int_1^2 \frac{1}{17}\,dx = \frac{1}{17}$. Also $1 + x^4 > x^4$ for $1 \le x \le 2$, so $\dfrac{1}{1+x^4} < \dfrac{1}{x^4}$ and

$\displaystyle\int_1^2 \frac{1}{1+x^4}\,dx < \int_1^2 x^{-4}\,dx = \left[\frac{x^{-3}}{-3}\right]_1^2 = -\frac{1}{24} + \frac{1}{3} = \frac{7}{24}$. Thus, we have the estimate

$\displaystyle\frac{1}{17} \le \int_1^2 \frac{1}{1+x^4}\,dx \le \frac{7}{24}$.

5. Such a function cannot exist. $f'(x) > 3$ for all x means that f is differentiable (and hence continuous) for all x. So by FTC2, $\int_1^4 f'(x)\,dx = f(4) - f(1) = 7 - (-1) = 8$. However, if $f'(x) > 3$ for all x, then $\int_1^4 f'(x)\,dx \ge 3\cdot(4-1) = 9$ by Comparison Property 8 in Section 4.2.

Another solution: By the Mean Value Theorem, there exists a number $c \in (1,4)$ such that

$f'(c) = \dfrac{f(4)-f(1)}{4-1} = \dfrac{7-(-1)}{3} = \dfrac{8}{3} \quad\Rightarrow\quad 8 = 3f'(c)$. But $f'(x) > 3 \quad\Rightarrow\quad 3f'(c) > 9$, so such a function cannot exist.

7. $f(x) = \displaystyle\int_0^{g(x)} \frac{1}{\sqrt{1+t^3}}\,dt$, where $g(x) = \int_0^{\cos x}\left[1 + \sin\left(t^2\right)\right]dt$. Using FTC1 and the Chain Rule (twice) we have

$f'(x) = \dfrac{1}{\sqrt{1+[g(x)]^3}}\,g'(x) = \dfrac{1}{\sqrt{1+[g(x)]^3}}\left[1 + \sin\left(\cos^2 x\right)\right](-\sin x)$. Now

$g\left(\frac{\pi}{2}\right) = \int_0^0\left[1+\sin\left(t^2\right)\right]dt = 0$, so $f'\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{1+0}}(1+\sin 0)(-1) = 1\cdot 1\cdot(-1) = -1$.

9. $f(x) = 2 + x - x^2 = (-x+2)(x+1) = 0 \quad\Leftrightarrow\quad x = 2$ or $x = -1$. $f(x) \ge 0$ for $x \in [-1, 2]$ and $f(x) < 0$ everywhere else. The integral $\int_a^b\left(2 + x - x^2\right)dx$ has a maximum on the interval where the integrand is positive, which is $[-1, 2]$. So $a = -1$, $b = 2$. (Any larger interval gives a smaller integral since $f(x) < 0$ outside $[-1, 2]$. Any smaller interval also gives a smaller integral since $f(x) \ge 0$ in $[-1, 2]$.)

11. (a) We can split the integral $\int_0^n [\![x]\!]\,dx$ into the sum $\sum\limits_{i=1}^{n}\left[\int_{i-1}^{i}[\![x]\!]\,dx\right]$. But on each of the intervals $[i-1, i)$ of integration, $[\![x]\!]$ is a constant function, namely $i - 1$. So the ith integral in the sum is equal to $(i-1)[i-(i-1)] = (i-1)$. So the original integral is equal to $\sum\limits_{i=1}^{n}(i-1) = \sum\limits_{i=1}^{n-1} i = \dfrac{(n-1)n}{2}$.

(b) We can write $\int_a^b [\![x]\!]\,dx = \int_0^b [\![x]\!]\,dx - \int_0^a [\![x]\!]\,dx$.

Now $\int_0^b [\![x]\!]\,dx = \int_0^{[\![b]\!]} [\![x]\!]\,dx + \int_{[\![b]\!]}^b [\![x]\!]\,dx$. The first of these integrals is equal to $\frac{1}{2}\left([\![b]\!]-1\right)[\![b]\!]$, by part (a), and since $[\![x]\!] = [\![b]\!]$ on $[[\![b]\!], b]$, the second integral is just $[\![b]\!]\left(b - [\![b]\!]\right)$. So

$\int_0^b [\![x]\!]\,dx = \frac{1}{2}([\![b]\!]-1)\,[\![b]\!] + [\![b]\!]\,(b - [\![b]\!]) = \frac{1}{2}\,[\![b]\!]\,(2b - [\![b]\!] - 1)$ and similarly

$\int_0^a [\![x]\!]\,dx = \frac{1}{2}\,[\![a]\!]\,(2a - [\![a]\!] - 1)$. Therefore, $\int_a^b [\![x]\!]\,dx = \frac{1}{2}\,[\![b]\!]\,(2b - [\![b]\!] - 1) - \frac{1}{2}\,[\![a]\!]\,(2a - [\![a]\!] - 1)$.

13. Differentiating the equation $\int_0^x f(t)\,dt = [f(x)]^2$ using FTC1 gives $f(x) = 2f(x)f'(x) \quad\Rightarrow$ $f(x)[2f'(x)-1] = 0$, so $f(x) = 0$ or $f'(x) = \frac{1}{2}$. $f'(x) = \frac{1}{2} \quad\Rightarrow\quad f(x) = \frac{1}{2}x + C$. To find C we substitute into the original equation to get $\int_0^x \left(\frac{1}{2}t + C\right) dt = \left(\frac{1}{2}x + C\right)^2 \quad\Leftrightarrow\quad \frac{1}{4}x^2 + Cx = \frac{1}{4}x^2 + Cx + C^2$. It follows that $C = 0$, so $f(x) = \frac{1}{2}x$. Therefore, $f(x) = 0$ or $f(x) = \frac{1}{2}x$.

15. Note that $\dfrac{d}{dx}\left(\displaystyle\int_0^x \left[\int_0^u f(t)\,dt\right] du\right) = \int_0^x f(t)\,dt$ by FTC1, while

$$\frac{d}{dx}\left[\int_0^x f(u)(x-u)\,du\right] = \frac{d}{dx}\left[x\int_0^x f(u)\,du\right] - \frac{d}{dx}\left[\int_0^x f(u)u\,du\right]$$

$$= \int_0^x f(u)\,du + xf(x) - f(x)x = \int_0^x f(u)\,du$$

Hence, $\int_0^x f(u)(x-u)\,du = \int_0^x \left[\int_0^u f(t)\,dt\right] du + C$. Setting $x = 0$ gives $C = 0$.

17. $\displaystyle\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\,\sqrt{n+1}} + \frac{1}{\sqrt{n}\,\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\,\sqrt{n+n}}\right)$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{n}{n+1}} + \sqrt{\frac{n}{n+2}} + \cdots + \sqrt{\frac{n}{n+n}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{1+1/n}} + \frac{1}{\sqrt{1+2/n}} + \cdots + \frac{1}{\sqrt{1+1}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right) \qquad \left[\text{where } f(x) = \frac{1}{\sqrt{1+x}}\right]$$

$$= \int_0^1 \frac{1}{\sqrt{1+x}}\,dx = \left[2\sqrt{1+x}\,\right]_0^1 = 2\left(\sqrt{2}-1\right)$$