8.3 THE INTEGRAL AND COMPARISON TESTS

A Click here for answers.

S Click here for solutions.

1. Use the Integral Test to determine whether the series

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \cdots$$

is convergent or divergent.

2–33 ▪ Determine whether the series is convergent or divergent.

2. $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n^{1.0001}}$

3. $\displaystyle\sum_{n=1}^{\infty} n^{-0.99}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt[3]{n}}$

5. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{n\sqrt{n}} + \frac{3}{n^3}\right)$

6. $\displaystyle\sum_{n=5}^{\infty} \frac{1}{(n-4)^2}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+3}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$

9. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

10. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2+1}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{1+n^2}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+2n+2}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+n^2}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{4^n+5}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n2^n}$

19. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

20. $\displaystyle\sum_{n=0}^{\infty} \frac{1+5^n}{4^n}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n(n+1)(n+2)}}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)2^n}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{3+\cos n}{3^n}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{5n}{2n^2-5}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+4}}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^4}$

30. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2-4}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+1}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n2^n}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-3n}{\sqrt[3]{n^{10}-4n^2}}$

8.3 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. Divergent
2. Convergent
3. Divergent
4. Divergent
5. Convergent
6. Convergent
7. Divergent
8. Divergent
9. Convergent
10. Convergent
11. Convergent
12. Convergent
13. Convergent
14. Convergent
15. Convergent
16. Converges
17. Converges
18. Converges
19. Diverges
20. Diverges
21. Converges
22. Converges
23. Converges
24. Diverges
25. Converges
26. Converges
27. Diverges
28. Converges
29. Converges
30. Converges
31. Converges
32. Converges
33. Converges

8.3 SOLUTIONS

E Click here for exercises.

1. $\frac{1}{3}+\frac{1}{7}+\frac{1}{11}+\frac{1}{15}+\cdots=\sum_{n=1}^{\infty}\frac{1}{4n-1}$.

The function $f(x)=\frac{1}{4x-1}$ is positive, continuous, and decreasing on $[1,\infty)$, so the Integral Test applies.

$$\int_1^\infty \frac{dx}{4x-1}=\lim_{b\to\infty}\int_1^b\frac{dx}{4x-1}=\lim_{b\to\infty}\left[\tfrac{1}{4}\ln(4x-1)\right]_1^b$$
$$=\lim_{b\to\infty}\left[\tfrac{1}{4}\ln(4b-1)-\tfrac{1}{4}\ln 3\right]=\infty$$

so the improper integral diverges, and so does the series.

2. $\sum_{n=5}^{\infty}\left(1/n^{1.0001}\right)$ is a p-series, $p=1.0001>1$, so it converges.

3. $\sum_{n=1}^{\infty}n^{-0.99}=\sum_{n=1}^{\infty}\left(1/n^{0.99}\right)$ which diverges since $p=0.99<1$.

4. $\sum_{n=1}^{\infty}\frac{2}{\sqrt[3]{n}}=2\sum_{n=1}^{\infty}\frac{1}{n^{1/3}}$, which is a p-series, $p=\frac{1}{3}<1$, so it diverges.

5. $\sum_{n=1}^{\infty}\left(\frac{2}{n\sqrt{n}}+\frac{3}{n^3}\right)=2\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}+3\sum_{n=1}^{\infty}\frac{1}{n^3}$, both of which are convergent p-series because $\frac{3}{2}>1$ and $3>1$, so $\sum_{n=1}^{\infty}\left(\frac{2}{n\sqrt{n}}+\frac{3}{n^3}\right)$ converges by Theorem 8 in Section 8.2.

6. $\sum_{n=5}^{\infty}\frac{1}{(n-4)^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p-series, $p=2>1$, so it converges.

7. $f(x)=\frac{1}{2x+3}$ is positive, continuous, and decreasing on $[1,\infty)$, so applying the Integral Test,

$$\int_1^\infty\frac{dx}{2x+3}=\lim_{t\to\infty}\left[\frac{1}{2}\ln(2x+3)\right]_1^t=\infty\quad\Rightarrow$$

$\sum_{n=1}^{\infty}\frac{1}{2n+3}$ is divergent.

8. Since $\frac{1}{\sqrt{x}+1}$ is continuous, positive, and decreasing on $[0,\infty)$ we can apply the Integral Test.

$$\int_1^\infty\frac{1}{\sqrt{x}+1}dx=\lim_{t\to\infty}\left[2\sqrt{x}-2\ln\left(\sqrt{x}+1\right)\right]_1^t$$

[using the substitution $u=\sqrt{x}+1$, so $dx=2(u-1)\,du$]

$$=\lim_{t\to\infty}\left(\left[2\sqrt{t}-2\ln\left(\sqrt{t}+1\right)\right]-(2-2\ln 2)\right)$$

Now $2\sqrt{t}-2\ln\left(\sqrt{t}+1\right)=2\ln\left(\frac{e^{\sqrt{t}}}{\sqrt{t}+1}\right)$ and so

$\lim_{t\to\infty}\left[2\sqrt{t}-2\ln\left(\sqrt{t}+1\right)\right]=\infty$ (using l'Hospital's Rule)

so both the integral and the original series diverge.

9. $f(x)=\frac{1}{x^2-1}$ is positive, continuous, and decreasing on $[2,\infty)$, so applying the Integral Test,

$$\int_2^\infty\frac{dx}{x^2-1}=\int_2^\infty\left(\frac{-1/2}{x+1}+\frac{1/2}{x-1}\right)dx$$
$$=\lim_{t\to\infty}\left[\ln\left(\frac{x-1}{x+1}\right)^{1/2}\right]_2^t=\ln\sqrt{3}\quad\Rightarrow$$

$\sum_{n=2}^{\infty}\frac{1}{n^2-1}$ converges.

10. $f(x)=xe^{-x^2}$ is continuous and positive on $[1,\infty)$, and since $f'(x)=e^{-x^2}\left(1-2x^2\right)<0$ for $x>1$, f is decreasing as well. Thus, we can use the Integral Test:

$\int_1^\infty xe^{-x^2}\,dx=\lim_{t\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_1^t=0-\left(-\frac{1}{2}e^{-1}\right)=\frac{1}{2e}$.

Since the integral converges, the series converges.

11. $f(x)=\frac{x}{2^x}$ is positive and continuous on $[1,\infty)$, and since $f'(x)=\frac{1-x\ln 2}{2^x}<0$ when $x>\frac{1}{\ln 2}\approx 1.44$, f is eventually decreasing, so we can apply the Integral Test. Integrating by parts, we get

$$\int_1^\infty\frac{x}{2^x}dx=\lim_{t\to\infty}\left(-\frac{1}{\ln 2}\left[\frac{x}{2^x}+\frac{1}{2^x\ln 2}\right]_1^t\right)$$
$$=\frac{1}{2\ln 2}+\frac{1}{2(\ln 2)^2}$$

since $\lim_{t\to\infty}\frac{t}{2^t}=0$ by l'Hospital's Rule, and so $\sum_{n=1}^{\infty}\frac{n}{2^n}$ converges.

12. $f(x)=\frac{1}{4x^2+1}$ is continuous, positive and decreasing on $[1,\infty)$, so applying the Integral Test,

$$\int_1^\infty\frac{dx}{4x^2+1}=\lim_{t\to\infty}\left[\frac{\arctan 2x}{2}\right]_1^t=\frac{\pi}{4}-\frac{\arctan 2}{2}<\infty,$$

so the series converges.

13. $f(x)=\frac{\arctan x}{1+x^2}$ is continuous and positive on $[1,\infty)$.

$f'(x)=\frac{1-2x\arctan x}{\left(1+x^2\right)^2}<0$ for $x>1$, since $2x\arctan x\geq\frac{\pi}{2}>1$ for $x\geq 1$. So f is decreasing and we can use the Integral Test.

$$\int_1^\infty\frac{\arctan x}{1+x^2}\,dx=\lim_{t\to\infty}\left[\tfrac{1}{2}(\arctan x)^2\right]_1^t$$
$$=\frac{(\pi/2)^2}{2}-\frac{(\pi/4)^2}{2}=\frac{3\pi^2}{32}$$

so the series converges.

14. $f(x) = \dfrac{\ln x}{x^2}$ is continuous and positive for $x \geq 2$, and $f'(x) = \dfrac{1 - 2\ln x}{x^3} < 0$ for $x \geq 2$, so f is decreasing.

$\displaystyle\int_2^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_2^t$ (by parts) $\overset{\text{H}}{=} 1$. Thus, $\displaystyle\sum_{n=1}^\infty \frac{\ln n}{n^2} = \sum_{n=2}^\infty \frac{\ln n}{n^2}$ converges by the Integral Test.

15. $f(x) = \dfrac{1}{x^2 + 2x + 2}$ is continuous and positive on $[1, \infty)$, and $f'(x) = -\dfrac{2x+2}{(x^2+2x+2)^2} < 0$ for $x \geq 1$, so f is decreasing and we can use the Integral Test.

$$\begin{aligned}\int_1^\infty \frac{1}{x^2+2x+2}\,dx &= \int_1^\infty \frac{1}{(x+1)^2+1}\,dx \\ &= \lim_{t\to\infty}\left[\arctan(x+1)\right]_1^t \\ &= \tfrac{\pi}{2} - \arctan 2\end{aligned}$$

so the series converges as well.

16. $\dfrac{1}{n^3+n^2} < \dfrac{1}{n^3}$ since $n^3 + n^2 > n^3$ for all n, and since $\displaystyle\sum_{n=1}^\infty \frac{1}{n^3}$ is a convergent p-series ($p = 3 > 1$), $\displaystyle\sum_{n=1}^\infty \frac{1}{n^3+n^2}$ converges also by the Comparison Test.

17. $\dfrac{3}{4^n+5} < \dfrac{3}{4^n}$ and $\displaystyle\sum_{n=1}^\infty \frac{3}{4^n}$ converges (geometric with $|r| = \dfrac{1}{4} < 1$) so by the Comparison Test, $\displaystyle\sum_{n=1}^\infty \frac{3}{4^n+5}$ converges also.

18. $\dfrac{3}{n2^n} \leq \dfrac{3}{2^n}$. $\displaystyle\sum_{n=1}^\infty \frac{3}{2^n}$ is a geometric series with $|r| = \dfrac{1}{2} < 1$, and hence converges, so $\displaystyle\sum_{n=1}^\infty \frac{3}{n2^n}$ converges also, by the Comparison Test.

19. $\dfrac{1}{\sqrt{n}-1} > \dfrac{1}{\sqrt{n}}$ and $\sum_{n=2}^\infty \dfrac{1}{\sqrt{n}}$ diverges (p-series with $p = \dfrac{1}{2} < 1$) so $\sum_{n=2}^\infty \dfrac{1}{\sqrt{n}-1}$ diverges by the Comparison Test.

20. $\dfrac{1+5^n}{4^n} > \dfrac{5^n}{4^n} = \left(\dfrac{5}{4}\right)^n$. $\sum_{n=0}^\infty \left(\dfrac{5}{4}\right)^n$ is a divergent geometric series ($|r| = \dfrac{5}{4} > 1$) so $\sum_{n=0}^\infty \dfrac{1+5^n}{4^n}$ diverges by the Comparison Test.

21. $\dfrac{\sin^2 n}{n\sqrt{n}} \leq \dfrac{1}{n\sqrt{n}} = \dfrac{1}{n^{3/2}}$ and $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{3/2}}$ converges ($p = \dfrac{3}{2} > 1$) so $\displaystyle\sum_{n=1}^\infty \frac{\sin^2 n}{n\sqrt{n}}$ converges by the Comparison Test.

22. $\dfrac{3}{n(n+3)} < \dfrac{3}{n^2}$. $\displaystyle\sum_{n=1}^\infty \frac{3}{n^2} = 3\sum_{n=1}^\infty \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$) so $\displaystyle\sum_{n=1}^\infty \frac{3}{n(n+3)}$ converges by the Comparison Test.

23. $\dfrac{1}{\sqrt{n(n+1)(n+2)}} < \dfrac{1}{\sqrt{n \cdot n \cdot n}} = \dfrac{1}{n^{3/2}}$ and since $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{3/2}}$ converges ($p = \dfrac{3}{2} > 1$), so does $\displaystyle\sum_{n=1}^\infty \frac{1}{\sqrt{n(n+1)(n+2)}}$ by the Comparison Test.

24. Use the Limit Comparison Test with $a_n = \dfrac{1}{\sqrt[3]{n(n+1)(n+2)}}$ and $b_n = \dfrac{1}{n}$.

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{n}{\sqrt[3]{n(n+1)(n+2)}} \\ &= \lim_{n\to\infty}\frac{1}{\sqrt[3]{1(1+1/n)(1+2/n)}} \\ &= 1 > 0\end{aligned}$$

so since $\displaystyle\sum_{n=1}^\infty \frac{1}{n}$ diverges, so does $\displaystyle\sum_{n=1}^\infty \frac{1}{\sqrt[3]{n(n+1)(n+2)}}$.

25. $\dfrac{n}{(n+1)2^n} < \dfrac{1}{2^n}$ and $\displaystyle\sum_{n=1}^\infty \frac{1}{2^n}$ is a convergent geometric series ($|r| = \frac{1}{2} < 1$), so $\displaystyle\sum_{n=1}^\infty \frac{n}{(n+1)2^n}$ converges by the Comparison Test.

26. $\dfrac{3+\cos n}{3^n} \leq \dfrac{4}{3^n}$ since $\cos n \leq 1$. $\displaystyle\sum_{n=1}^\infty \frac{4}{3^n}$ is a geometric series with $|r| = \frac{1}{3} < 1$ so it converges, and so $\displaystyle\sum_{n=1}^\infty \frac{3+\cos n}{3^n}$ converges by the Comparison Test.

27. $\dfrac{5n}{2n^2-5} > \dfrac{5n}{2n^2} = \dfrac{5}{2}\left(\dfrac{1}{n}\right)$ and since $\dfrac{5}{2}\displaystyle\sum_{n=1}^\infty \frac{1}{n}$ diverges (harmonic series) so does $\displaystyle\sum_{n=1}^\infty \frac{5n}{2n^2-5}$ by the Comparison Test.

28. $\dfrac{n}{\sqrt{n^5+4}} < \dfrac{n}{\sqrt{n^5}} = \dfrac{1}{n^{3/2}}$. $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{3/2}}$ is a convergent p-series ($p = \frac{3}{2} > 1$) so $\displaystyle\sum_{n=1}^\infty \frac{n}{\sqrt{n^5+4}}$ converges by the Comparison Test.

29. $\dfrac{\arctan n}{n^4} < \dfrac{\pi/2}{n^4}$ and $\dfrac{\pi}{2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$ converges ($p = 4 > 1$) so $\displaystyle\sum_{n=1}^{\infty}\frac{\arctan n}{n^4}$ converges by the Comparison Test.

30. Use the Limit Comparison Test with $a_n = \dfrac{1}{n^2-4}$ and $b_n = \dfrac{1}{n^2}$: $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^2}{n^2-4} = 1 > 0$. Since $\sum_{n=3}^{\infty} b_n$ converges ($p = 2 > 1$), $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n^2-4}$ also converges.

31. Let $a_n = \dfrac{n^2+1}{n^4+1}$ and $b_n = \dfrac{1}{n^2}$. Then $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^4+n^2}{n^4+1} = 1 > 0$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$), so is $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+1}{n^4+1}$ by the Limit Comparison Test.

32. Let $a_n = \dfrac{n+1}{n2^n}$ and $b_n = \dfrac{1}{2^n}$. Then $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n+1}{n} = 1 > 0$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n}$ is a convergent geometric series ($|r| = \frac{1}{2} < 1$), $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n2^n}$ converges by the Limit Comparison Test.

33. Use the Limit Comparison Test with $a_n = \dfrac{n^2-3n}{\sqrt[3]{n^{10}-4n^2}}$ and $b_n = \dfrac{1}{n^{4/3}}$.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^{10/3}-3n^{7/3}}{\sqrt[3]{n^{10}-4n^2}} = \lim_{n\to\infty}\frac{1-3/n}{\sqrt[3]{1-4n^{-8}}} = 1 > 0$$

so since $\sum_{n=1}^{\infty} b_n$ converges ($p = \frac{4}{3} > 1$), so does $\displaystyle\sum_{n=1}^{\infty}\frac{n^2-3n}{\sqrt[3]{n^{10}-4n^2}}$.